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CYCLES THROUGH SPECIFIED VERTICES OF A GRAPH

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We prove that if S is a set of $k-1$ vertices in a k-connected graph G, then the cycles through S generate the cycle space of G. Moreover, when $k \geqq 3$, each cycle of G can be expressed as the sum of an odd number of cycles through S. On the other hand, if S is a set of k vertices, these conclusions do not necessarily hold, and we characterize the exceptional cases. As corollaries, we establish the existence of odd and even cycles through specified vertices and deduce the existence of long odd and even cycles in graphs of high connectivity.

1. Introduction

A well-known theorem of Dirac [2] states that if e and f are two edges of a k-connected graph G ($k \geqq 2$), and if S is a set of $k-2$ vertices of G, then G contains a cycle which includes e, f, and every vertex of S. A number of papers on this theme can be found in the literature: Dirac and Thomassen [3], Holton, McKay and Plummer [4], Kinney and Alexander [5], Lick [6], Plummer [11], Thomassen [12], Watkins and Mesner [16], Wilson, Hemminger and Plummer [17], Woodall [18]. Several related questions have also been posed. We consider one, due to Toft [13].

For $n \geqq 2$, let $f(n)$ denote the least integer for which the following statement is true: if a graph G is $f(n)$-connected and contains at least one odd cycle, then for any n vertices of G there is an odd cycle of G containing these n vertices. What is the value of $f(n)$?

Toft proved that $n+1 \leqq f(n) \leqq 2n-1$. The lower bound arises from a simple construction. Let G_1 be a nonempty graph on n vertices, let G_2 be an empty graph (no edges) on $m \geqq n$ vertices, and let S be a set of n vertices in G_2. Denote by G the join $G_1 \vee G_2$ of G_1 and G_2. Then G is n-connected, and contains an odd cycle. However, no odd cycle passes through every vertex of S.

Later, Lovász [8] proposed the conjecture that $f(n)=n+1$ and also a somewhat stronger conjecture concerning the cycle space of a graph. In this paper, we prove these conjectures, settle the analogue of Toft's problem for even cycles, and establish the existence of long odd and even cycles in graphs of high connectivity.

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2. The cycle space

It is well-known that the cycle space of a graph has a basis which consists only of cycles. A set $\mathscr{C}$ of cycles in a graph G is said to *generate* the cycle space of G if it contains such a basis or, equivalently, if every cycle C of G can be written as

$$C=\sum_{i=1}^{N} C_i \quad (C_i \in \mathscr{C})$$

where a cycle is regarded as the set of its edges, and addition is modulo 2 (so that $\sum_{i=1}^{N} C_i$ is the set of those edges of G which are contained in an odd number of cycles C_i).

The graphs which we consider in this paper have no loops but may have multiple edges. We shall prove two theorems.

Theorem 1. *Let* $S=\{u, v\}$ *be a set of two vertices in a 2-connected graph* G. *Then exactly one of the following two statements is true:*

(i) *the cycles through* S *generate the cycle space of* G;

(ii) G *contains a connected subgraph* H, *disjoint from* S, *with two bridges (in the sense of Tutte* [14]*)* B_x *(*$x=u$ *or* v*) such that* B_x *contains* x *and has exactly two vertices of attachment to* H.

The structure of the graphs described in condition (ii) of Theorem 1 is illustrated in Figure 1.

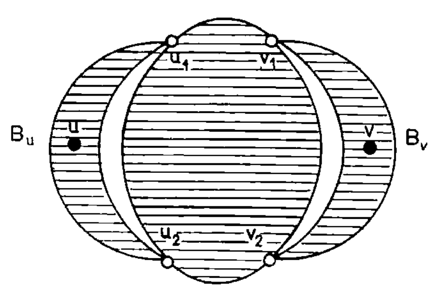

Fig. 1

Theorem 2. *Let* S *be a set of* k *vertices in a* k*-connected graph* G, *where* $k \geqq 3$. *Then one of the following three statements is true:*

(i) *every cycle in* G *can be expressed as the sum of an odd number of cycles through* S;

(ii) G *contains a set* X *of* k *vertices, disjoint from* S, *such that* X *(regarded as a subgraph of* G*) has at least* $k+1$ *bridges,* k *of which each contain one vertex of* S;

(iii) $k=3$ *and* G *contains adjacent vertices* x_0, x_1 *such that* $G-\{x_0, x_1\}$ *is a union of four connected subgraphs* H, G_0, G_1, G_2, *where* H *includes no vertex of* S, *each* G_i *includes just one vertex of* S, H *and* G_i *have exactly one vertex in common* $(i=0, 1, 2)$ *and* G_i *and* G_j *are disjoint* $(i, j=0, 1, 2)$.

The structures of the graphs described in conditions (ii) and (iii) of Theorem 2 are illustrated in Figures 2(a) and 2(b), respectively.

Remark 1. The conditions in Theorem 2 are not mutually exclusive. However, no graph which satisfies (ii) or (iii) can also satisfy (i) because, in both cases, there exists an edge e which lies in no cycle through S: in (ii), take e to be any edge in a bridge of X which contains no vertex of S; in (iii), take e to be the edge x_0x_1.

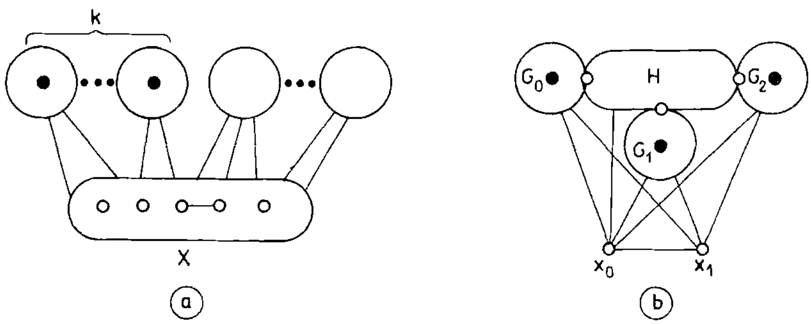

Fig. 2

Remark 2. It follows from Menger's theorem (and also from our proof of Theorem 2) that, in cases (ii) and (iii) of Theorem 2, G contains a subdivision of the complete bipartite graph $K_{k,k}$ in which one colour-class is the set S.

One easy corollary of Theorem 2 is a result similar to, but neither weaker nor stronger than, Dirac's theorem.

Corollary 1. *Let S be a set of k vertices in a k-connected graph G, where $k \geqq 4$, and let e be an edge of G. Then exactly one of the following two statements is true:*

(i) *there exists a cycle in G which includes e and every vertex of S;*

(ii) *G contains a set X of k vertices such that the elements of $S \cup \{e\}$ belong to distinct bridges of X.*

To see this, we note that, since G is 2-connected, e lies in some cycle C of G. If case (i) of the theorem applies, C can be expressed as the sum of an odd number of cycles through S. Clearly, one of these cycles must include e. If (ii) applies, it suffices to consider the case in which e belongs to the same bridge as some element of S. By Remark 2, G contains a subdivision H of a complete bipartite graph $K_{k,k}$. If e is an edge of H, there is clearly a cycle in G which includes e and every vertex of S. Otherwise, by Menger's theorem, its ends are connected to H by two disjoint paths. Since e belongs to the same bridge as some element of S, we may assume that at least one of these paths does not end at a vertex of X; again, there is a cycle which includes e and every vertex of S.

From Corollary 1, we can immediately deduce a theorem of Watkins and Mesner [16].

Corollary 2. *Let S be a set of $k+1$ vertices in a k-connected graph G, where $k \geqq 4$. Then exactly one of the following two statements is true:*

(i) *there exists a cycle in G which includes every vertex of S;*

(ii) *G contains a set X of k vertices, disjoint from S, such that the vertices of S belong to distinct bridges of X.*

For, let $v \in S$, let e be an edge incident with v, and set $S' = S \setminus \{v\}$. If there exists a cycle in G which includes e and every vertex of S', then (i) holds. Otherwise, by Corollary 1, G contains a set X of k vertices such that the elements of $S \cup \{e\}$ belong to distinct bridges of X. If $v \in X$, then it follows from Remark 2 that there exists a cycle in G which includes every vertex of S. If $v \notin X$, then the vertices of S belong to distinct bridges of X, and (ii) holds.

Remark 3. Watkins and Mesner did, in fact, prove this result for $k \geqq 3$. They also obtained an analogous theorem for the case of 2-connected graphs. Neither these results, nor the analogues of Corollary 1 for 2-connected and 3-connected graphs can be readily deduced from Theorems 1 and 2.

A more significant consequence of Theorems 1 and 2 is the following result, conjectured originally by Lovász [8]. We state it in two equivalent forms.

Corollary 3. *Let S be a set of $k-1$ vertices in a k-connected graph G. Then the cycles through S generate the cycle space of G.*

Corollary 3′. *Let S be a set of $k-1$ vertices in a k-connected graph G, and let F be a set of edges of G which do not form a coboundary. Then there exists a cycle C_i through S such that $|F \cap C_i|$ is odd.*

To see that Corollary 3 implies Corollary 3′, consider S, G and F as in Corollary 3′. Since F is not a coboundary, there exists a cycle C in G such that $|F \cap C|$ is odd. By Corollary 3, there exist cycles $C_1, \ldots, C_N$ through S such that

$$C = \sum_{i=1}^{N} C_i.$$

Then

$$|F \cap C| \equiv \sum_{i=1}^{N} |F \cap C_i| \pmod 2$$

which implies that $|F \cap C_i|$ is odd for at least one C_i.

The solution to Toft's problem follows directly from Corollary 3′ on taking F to be the set of all edges of G. (F is not a coboundary because, by assumption, G is not bipartite.)

The analogue of Toft's problem for even cycles can also be solved by our methods.

Consider, first, the case of 2-connected graphs. Such graphs may contain vertices which lie on no even cycles at all. For instance, if H is a 2-connected bipartite graph and G is the graph obtained from H by adding a vertex v and joining v to one vertex in each colour-class of H, then G is 2-connected, but the vertex v lies on no even cycle.

To deal with k-connected graphs, where $k \geqq 3$, we apply Theorem 2. Let S be a set of k vertices in a k-connected graph G, where $k \geqq 3$, and let C be an even cycle in G. If C can be expressed as the sum of an odd number of cycles through S, one of these cycles must clearly be even. Thus, in this case, G contains an even cycle through S. If C cannot be so expressed, statement (ii) of Theorem 2 applies and, by Remark 2, G contains a subdivision H of a complete bipartite graph $K_{k,k}$ in which one colour-class is the set S. Let C be an even cycle in H. Now, by Theorem 2, every cycle in the complete bipartite graph can be expressed as the sum of an odd number of cycles through S. It follows that C can be expressed as the sum of an odd number of cycles in H through S. As before, one of these cycles must be even.

Thus we have

Corollary 4.

(*a*) *Let S be a set of $k-1$ vertices in a k-connected nonbipartite graph G. Then there exists an odd cycle in G through every vertex of S.*

(*b*) *Let S be a set of k vertices in a k-connected graph G, where $k \geqq 3$. Then there exists an even cycle in G through every vertex of S.*

Remark 4. Mamoun [9] has independently proved Corollary 4(a) and a slightly weaker form of Corollary 4(b).

Finally, we deduce from Corollary 4 a result about long odd and even cycles in k-connected graphs.

Corollary 5. *Let G be a k-connected nonhamiltonian graph, where $k \geqq 2$. Then there exist in G an odd cycle of length at least $2k-1$ (provided that G is not bipartite) and an even cycle of length at least $2k$.*

To see this, we observe that, by a theorem of Chvátal and Erdős [1], G contains an independent set of k vertices. Thus the set S in Corollary 4 can be chosen to be independent, and the resulting cycles satisfy the requirements of Corollary 5. (The one case omitted here, that of an even cycle of length at least four in a 2-connected nonhamiltonian graph, is trivial.)

Remark 5. There is a stronger form of Corollary 5, due to Voss and Zuluaga [15]. They prove that if G is a 2-connected nonhamiltonian graph in which each degree is at least k, then the same conclusion holds.

Our proofs of both theorems make extensive use of the following result, due to Perfect [10]. Let G be a graph, let X be a set of vertices of G, and let $v \in V(G) \setminus X$. By a (v, X)-*path* we mean a path which connects v to some vertex of X and has no other vertex in common with X. Two (v, X)-paths are *independent* if they have no vertex in common but v.

Lemma. *Let G be a graph, let X be a set of vertices of G, and let $v \in V(G) \setminus X$. Suppose that there exist k independent (v, X)-paths. Suppose, also, that there exist k' independent (v, X)-paths (not necessarily chosen from the previously-defined set) linking v to the set $Y \subseteq X$, where $k' \leqq k$. Then there exist k independent (v, X)-paths, k' of which link v to Y.*

3. Proof of Theorem 1

We define a cycle C in G to be of *type m* if it includes precisely m vertices of S ($m=0, 1, 2$). The theorem will be proved by demonstrating that each cycle of type m ($m=0, 1$) can be expressed as a sum of cycles whose types are greater than m, unless statement (ii) of the theorem applies.

Let C be a cycle of type 0 or 1. Without loss of generality, we may assume that v does not lie on C. We fix an orientation of C and denote by $C[x, y]$ the segment of C between vertices x and y. If v is connected to x, y by independent (v, C)-paths P_x, P_y, respectively, we set

$$C_{x,y} = P_x \cup P_y \cup C[y, x].$$

Suppose, first, that C is of type 0. By Menger's theorem, v is connected by independent paths P_x, P_y to two vertices x, y of C. Then

$$C = C_{x,y} + C_{y,x} \tag{1}$$

expresses C as the sum of two cycles of type 1 or type 2.

Now, suppose that C is of type 1. If v is connected by independent paths P_x, P_y, P_z to three vertices x, y, z, respectively, on C (occurring in this cyclic order), we may assume, without loss of generality, that $u \in C[z, x]$. Then

$$C = C_{x,y} + C_{y,z} + C_{x,z}$$

expresses C as a sum of cycles of type 2. Otherwise, by Menger's theorem, v is connected by independent paths P_x, P_y to two vertices x, y of C and there exist vertices v_1 on P_x and v_2 on P_y which separate v from C. If $u \in \{x, y\}$, equation (1) expresses C as the sum of two cycles of type 2. If not, we may assume, without loss of generality, that $u \in C[x, y]$, as in Figure 3(a).

We now consider paths from u to the cycle $C_{x,y}$. Suppose that u is connected to $C_{x,y}$ by three independent paths Q_x, Q_y, Q_z. By the Lemma, we may assume that Q_x terminates at x and Q_y terminates at y. Let Q_z terminate at z.

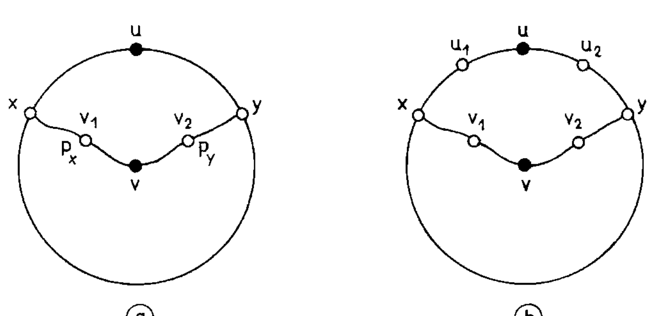

Fig. 3

If $z \in C[y, x]$, set

$$C_1 = P_x \cup P_y \cup Q_y \cup Q_x$$
$$C_2 = P_x \cup P_y \cup C[y, z] \cup Q_z \cup Q_x$$
$$C_3 = P_x \cup P_y \cup Q_y \cup Q_z \cup C[z, x]$$

Otherwise, we may assume, without loss of generality, that $z \in P_x$. In this case, set

$$C_1 = P_x \cup P_y \cup Q_y \cup Q_x$$
$$C_2 = P_x[z, v] \cup P_y \cup Q_y \cup Q_z$$
$$C_3 = P_x[z, v] \cup P_y \cup C[y, x] \cup Q_x \cup Q_z.$$

In both cases $C = C_1 + C_2 + C_3 + C_{x,y}$ expresses C as a sum of cycles of type 2.

If u is not connected to $C_{x,y}$ by three independent paths, it follows from Menger's theorem that there exist vertices $u_1 \in C[x, u]$ and $u_2 \in C[u, y]$ which separate u from $C_{x,y}$, as in Figure 3(b). Thus G is as described in statement (ii) of the theorem.

We have shown that if (i) does not hold, then (ii) does. To see that both statements cannot hold simultaneously, consider a graph G satisfying (ii), and let C be a cycle in G consisting of a (u_1, u_2)-path in B_u followed by a (u_2, u_1)-path in H.

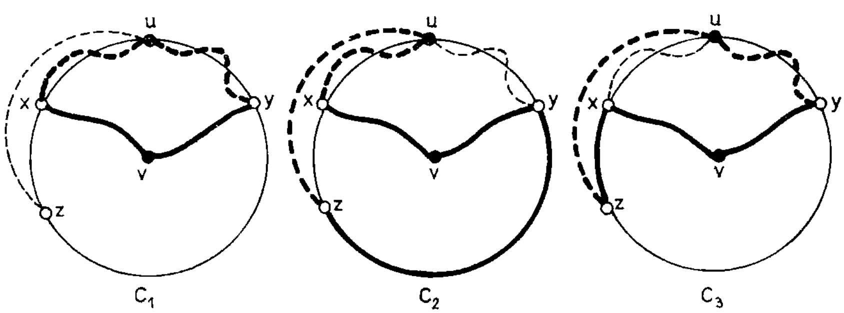

Fig. 4

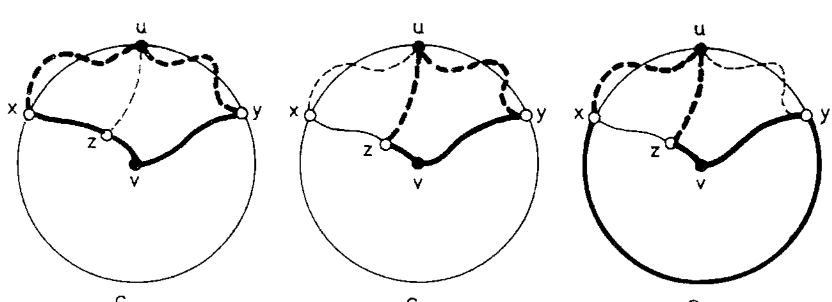

Fig. 5

Suppose that C can be expressed as the sum of N cycles through $S=\{u, v\}$. Because each such cycle includes exactly one edge of B_u incident to u_1 and the same is true of C, N must be odd. However, N must also be even because each cycle through S includes exactly one edge of B_v incident to v_1 and C includes no such edge. This contradiction proves our assertion.

4. Proof of theorem 2

This proof proceeds along the same lines as the proof of Theorem 1. Let $\mathscr{C}$ denote the set of cycles through S, and let C be a cycle not in $\mathscr{C}$. Set $T=V(C)\cap S$. We define C to be of *type* (m, n) if

(i) $|T|=m$;

(ii) there exist a vertex $v\in S\setminus T$ and n independent (v, C)-paths ending at vertices of T.

By convention, each cycle in $\mathscr{C}$ has type $(k, 0)$. We shall prove the theorem by demonstrating that each cycle of type (m, n), where $m\leqq k-1$, can be expressed as the sum of an odd number of cycles whose types are lexicographically greater than (m, n), unless either statement (ii) or (iii) of the theorem applies.

Let C be a cycle of type (m, n), where $m\leqq k-1$, let $T=V(C)\cap S$, and let $v\in S\setminus T$ as in (ii) above. If C has length l then, by the Lemma, v is linked by independent (v, C)-paths to a set X of min $\{k, l\}$ vertices of C such that $T'=X\cap T$ has n elements. For $x\in X$, let P_x denote the path linking v and x. As before, set $C_{x,y}=P_x\cup P_y\cup C[y, x]$.

We first dispose of two special cases.

(A) Suppose that $l=2$. Let $C=xeyfx$, and let H denote the underlying simple subgraph of G (obtained by deleting all but one edge between each pair of vertices). Since $l=2$, there are just two independent (v, C)-paths, P_x and P_y, in H. However, because H is k-connected and $k\geqq 3$, we may assume the existence of a second (v, y)-path P_0, independent of P_x and internally-disjoint from P_y (Figure 6(a)).

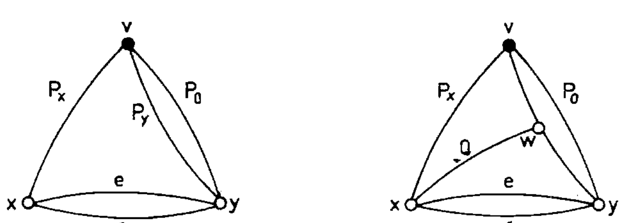

Fig. 6

Since H is simple, the subgraph $P_y\cup P_0$ contains at least three vertices. By the Lemma, there are three independent paths from x to $P_y\cup P_0$, two of which terminate at y and v and may be assumed to be the edge e and the path P_x, respectively. We may also assume, without loss of generality, that the third path Q

terminates at an internal vertex w of P_y (Figure 6(b)). Let

$$C_1 = P_y[w, y] \cup Q \cup P_x \cup P_0$$
$$C_2 = P_y[v, w] \cup Q \cup e \cup P_0$$
$$C_3 = P_y \cup f \cup P_x.$$

Then (see Figure 7) $C = C_1 + C_2 + C_3$.

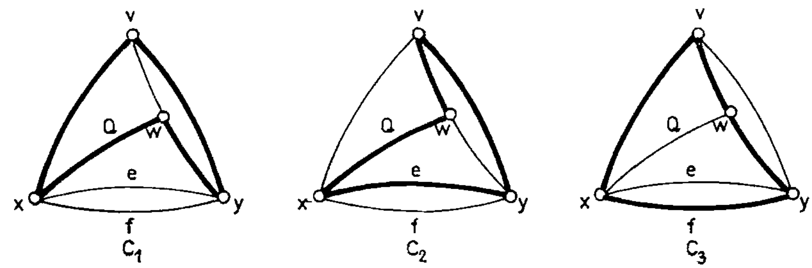

Fig. 7

We shall henceforth assume G to be simple. Since the case $l=2$ has been dealt with, this results in no loss of generality.

(B) Suppose that $l=m=n$. Then $V(C)=T=T'$. Let $C=12\ldots m1$. Since $m+1 \leqq k$ and G is k-connected, we may assume, without loss of generality, that v is connected by internally-disjoint paths $P_0, P_1, \ldots, P_m$ to $1, 1, 2, 3, \ldots, m$, respectively (Figure 8(a)).

Consider the subgraph

$$F = \left(C \cup \left(\bigcup_{i=0}^{m} P_i\right)\right) - \{m\}.$$

Since G is simple, F has at least $m+1$ vertices. Therefore, by the Lemma, there exist $m+1$ independent (m, F)-paths, three of which terminate at vertices v, 1 and

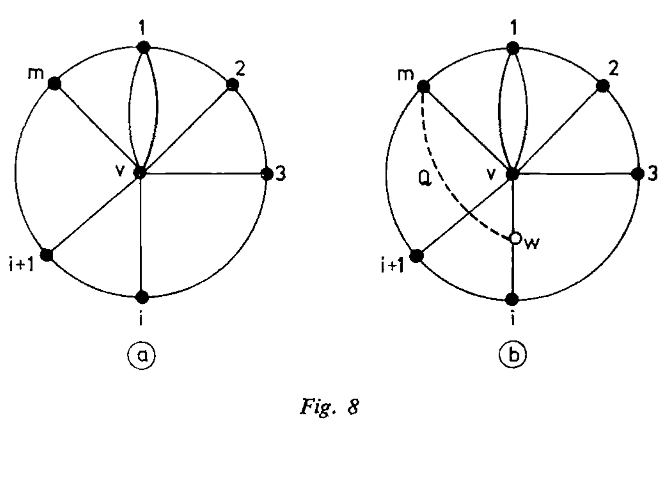

Fig. 8

$m-1$ (and may be assumed to be the path P_m and the edges $m1$ and $m(m-1)$, respectively) and one of which, Q say, terminates at an internal vertex w of some P_i $(0 \leqq i \leqq m-1)$. Since both P_0 and P_1 are $(v, 1)$-paths, we may assume that w is an internal vertex of some P_i $(1 \leqq i \leqq m-1)$. (Figure 8(b)). Let

$$C_1 = C_{i,i+1}$$
$$C_2 = C[1, m] \cup Q \cup P_i[v, w] \cup P_0$$
$$C_3 = C[1, i] \cup P_i[w, i] \cup Q \cup C[i+1, m] \cup P_{i+1} \cup P_0.$$

Then (Figure 9) $C = C_1 + C_2 + C_3$.

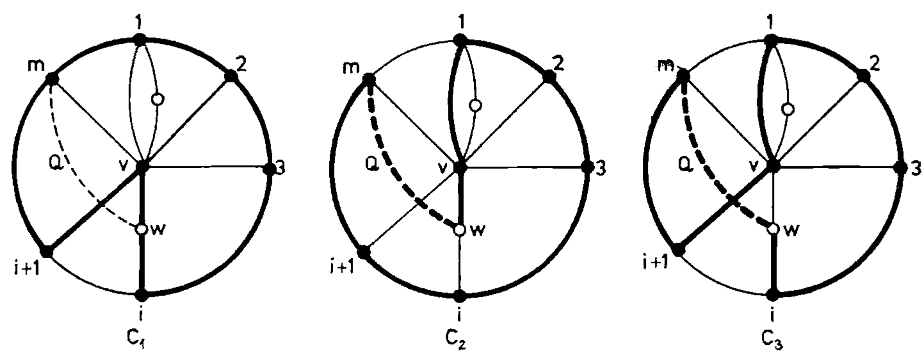

Fig. 9

We may now assume that $l \geqq 3$ and $l \geqq n+1$. Since $k \geqq 3$ and $k \geqq m+1 \geqq n+1$, we have

(2) $$|X| = \min\{k, l\} \geqq \max\{3, n+1\}.$$

Let the vertices of T, in cyclic order, be the integers modulo m, and set

$$X_i = X \cap C[i, i+1], \quad 1 \leqq i \leqq m, \quad \text{if} \quad T \neq \emptyset$$
$$X_0 = X, \qquad \text{if} \quad T = \emptyset.$$

(C) Suppose that some X_i contains three vertices x, y, z (occurring in this order on C). Then $C = C_{x,y} + C_{y,z} + C_{x,z}$. So we may assume that

(3) $$|X_i| \leqq 2, \quad 1 \leqq i \leqq m.$$

Since $|X| \geqq 3$, it follows from (3) that

(4) $$m \geqq 2.$$

By (2,) (3) and (4), we have

$$2m \geqq \sum_{i=1}^{m} |X_i| = 2|T'| + |X \setminus T'| = |X| + n \geqq 2n+1.$$

Thus $m>n$. We deduce that $T \not\subseteq X$ and hence $l>|X|$. So

(5) $$|X| = \min\{k, l\} = k.$$

(**D**) We now establish the existence of a vertex $j \in T \setminus T'$ such that $|X_{j-1} \cup X_j| \geqq 3$. If there is no such vertex, then

(6) $$\sum_{i \in T \setminus T'} |X_{i-1} \cup X_i| \leqq 2|T \setminus T'| = 2(m-n).$$

Also, by (3)

(7) $$\sum_{i \in T'} |X_{i-1} \cup X_i| \leqq 3|T'| = 3n.$$

Combining (6) and (7), we obtain

(8) $$\sum_{i \in T} |X_{i-1} \cup X_i| \leqq 2m+n.$$

However, using (5) we see that

(9) $$\sum_{i \in T} |X_{i-1} \cup X_i| = 3|T'|+2|X \setminus T'| = 2k+n$$

because each vertex of T' is counted three times in this sum, and each vertex of $X \setminus T'$ is counted twice. Now (8) and (9) yield $m \geqq k$, a contradiction.

By (3) and (D) we may assume, without loss of generality, that $1 \in T \setminus T'$ and $|X_1|=2$, $|X_m| \geqq 1$. Let a denote the vertex of X immediately preceding 1 on C, and let b and c denote the vertices of X immediately succeeding 1 on C. Set $G' = \bigcup_{x \in X} P_x \cup C[b, a]$. Then (again by the Lemma) 1 is linked by independent $(1, G')$-paths to a set Y of k vertices of G', where $\{a, b\} \subseteq Y$. For $y \in Y$, let Q_y denote the path linking 1 and y, and, for $1 \leqq i \leqq m$, set $Y_i = Y \cap C[i, i+1]$.

Because $C = C_{b,c} + C_{c,b}$ and $C_{b,c}$ is of type at least $(m+1, 0)$, it will suffice to demonstrate that $C_{c,b}$ can be expressed as the sum of an even number of cycles of types greater than that of C. We do so in most of the cases to be considered below and, for this reason, set $D = C_{c,b}$.

(**E**) Suppose that some Y_i ($i \neq 1, m$) contains two vertices x, y (occurring in this order on C, as in Figure 10).

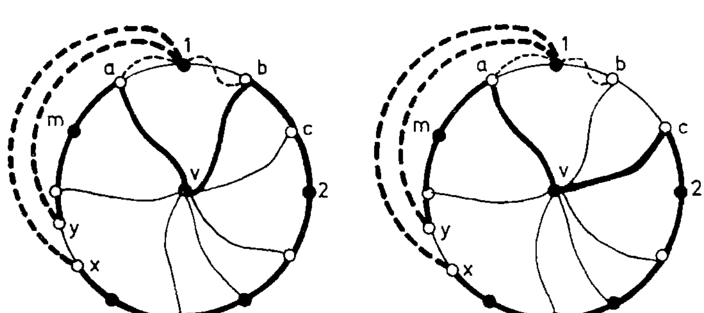

Fig. 10

Set

$$L = C[c, x] \cup Q_x \cup Q_y \cup C[y, a] \cup P_a.$$

Then $D = (L \cup P_b \cup C[b, c]) + (L \cup P_c)$.

(F) Suppose that $y \in Y$ is either v or an internal vertex of some P_x, $x \neq b, c$ (Figure 11). Set $L = C[c, a] \cup Q_a \cup Q_y \cup P_x[v, y]$.

Then $D = (L \cup P_b \cup C[b, c]) + (L \cup P_c)$.

(G) Suppose that $y \in Y$ is a vertex of $C[m, a]$ different from a (Figure 12).

Set $L = C[c, y] \cup Q_y \cup Q_a \cup P_a$.

Then $D = (L \cup P_b \cup C[b, c]) + (L \cup P_c)$.

(H) Suppose that $y \in Y$ is a vertex of $C[c, 2]$ different from c (Figure 13).

Set $L = Q_b \cup Q_y \cup C[y, a] \cup P_a$.

Then $D = (L \cup P_b) + (L \cup P_c \cup C[b, c])$.

(I) Suppose that $y \in Y$ is an internal vertex of P_c (Figure 14).

Set $L = C[c, a] \cup P_a \cup Q_b \cup Q_y$.

Then $D = (L \cup P_b \cup P_c[y, c]) + (L \cup P_c[v, y] \cup C[b, c])$.

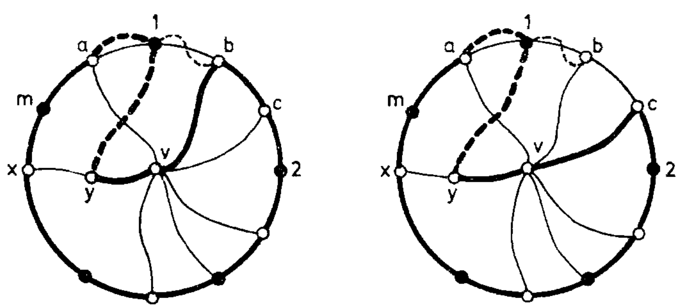

Fig. 11

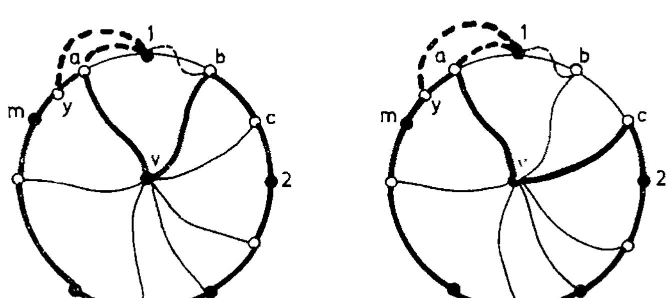

Fig. 12

(J) Suppose that $y \in Y$ is an internal vertex of P_b (Figure 15).

Set $C' = Q_a \cup Q_b \cup C[b, a]$.

Then (on deleting the internal vertices of $P_b[y, b]$) we see that C' is of type at least $(m, n+1)$, and $D = C' + C'_{b,c}$.

We may now assume that none of the cases (C)—(J) occurs. Thus

$$Y = \bigcup_{i \neq 1, m} Y_i \cup \{a\} \cup (Y \cap C[b, c])$$

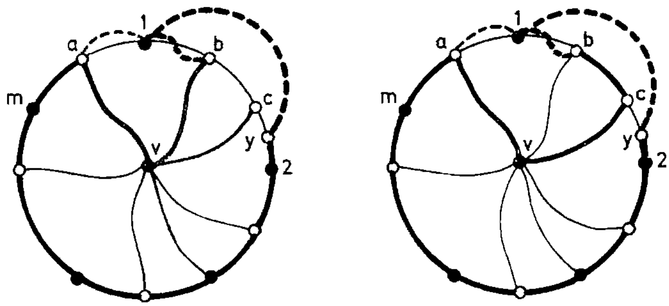

Fig. 13

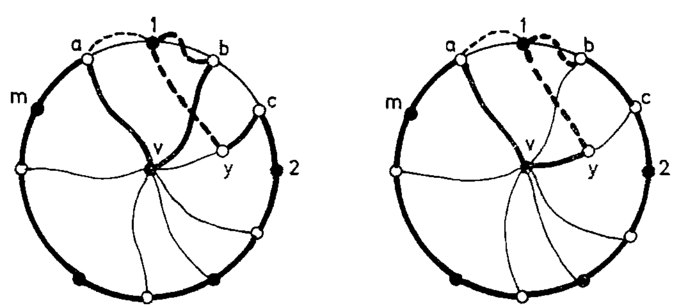

Fig. 14

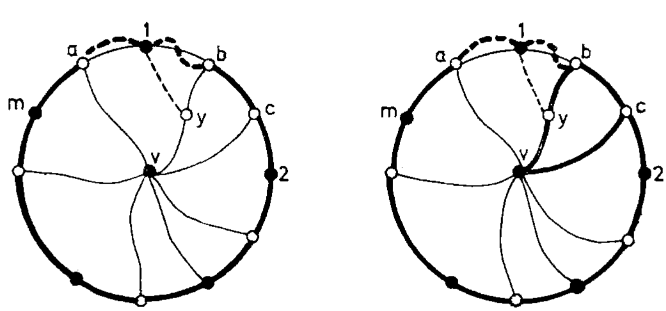

Fig. 15

and so

$$k = |Y| \leqq \sum_{i \neq 1, m} |Y_i| + 1 + |Y \cap C[b, c]| \leqq k - 2 + |Y \cap C[b, c]|.$$

Consequently, some $y \in Y$ is a vertex of $C[b, c]$ different from b.

(K) Suppose that some $y \in Y$ is an internal vertex of $C[b, c]$. (Figure 16).

Set $L = C[c, a] \cup Q_a \cup Q_y$.

Then $D = (L \cup C[y, c]) + (L \cup C[b, y] \cup P_b \cup P_c)$.

(L) Suppose that there exists $i \neq 1$ with $|X_i| = 2$ (Figure 17).

Set $\hat{C} = Q_a \cup Q_c \cup C[c, a]$. Then (on deleting the internal vertices of $C[b, c]$) we see that $\hat{C}$ is of type at least $(m, n+1)$. Let $X_i = \{d, e\}$ (in this order on C). Then

$$C = \hat{C} + \hat{C}_{d,e} + C_{d,e}.$$

In view of (L), we may assume that $|X_i| \leqq 1$, $i \neq 1$. Therefore

$$m+1 \geqq \sum_{i=1}^{m} |X_i| = |X| + n = k + n \geqq m + n + 1.$$

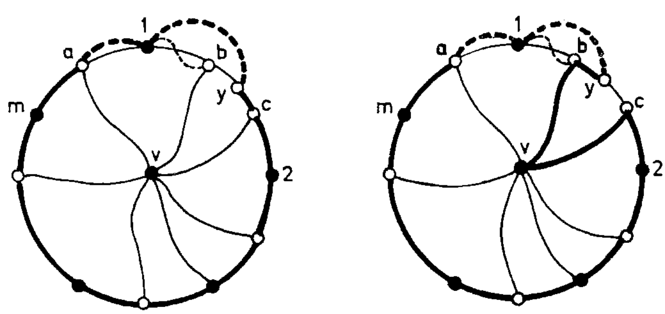

Fig. 16

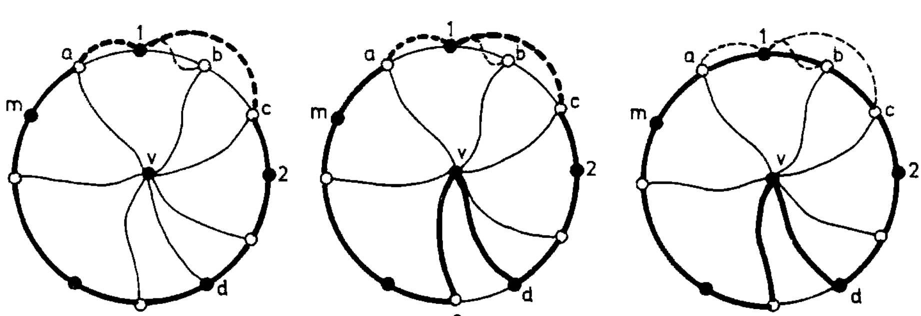

Fig. 17

We deduce that $n=0, m=k-1, S=\{1, 2, \dots, k-1, v\}$ and

$$|X_i| = \begin{cases} 2 & i = 1 \\ 1 & i \neq 1. \end{cases}$$

Let $X_2=\{d\}$, and set

$$G'' = \bigcup_{x\in X} P_x \bigcup_{y\in Y} Q_y \cup C'[d, c].$$

Then (by the Lemma) 2 is linked by independent $(2, G'')$-paths to a set Z of k vertices of G'', where $\{c, d\}\subseteq Z$. For $z\in Z$, let R_z denote the path linking 2 and z, and set $C''=R_c\cup R_d\cup C'[d, c]$. Note that neither 1 nor any internal vertex of any Q_y $(y\neq a, b)$ belongs to Z since we have eliminated case (H). Thus, we may assume (by the same argument as was used on Y) that

(10) $$Z\cap C[b, c] = Y\cap C[b, c] = \{b, c\}.$$

Set

$$Z_i = Z\cap C''[i, i+1], \quad 1 \leqq i \leqq k-1.$$

Then it follows from (E) and (10) that

$$X_1 = Y_1 = Z_1 = \{b, c\} \quad \text{and} \quad |Y_i| = |Z_i| = 1, \quad i \neq 1.$$

We now relabel v, a, b, c, d so that

$$S = \{0, 1, \dots, k-1\}$$

$$X_1 = Y_1 = Z_1 = \{x_0, x_1\}$$

and, for $2\leqq i\leqq k-1$,

$$X_i = \{x_i\}, \quad Y_i = \{y_i\}, \quad Z_i = \{z_i\}.$$

Let P_i, Q_i and R_i denote the paths from 0 to x_i, 1 to y_i and 2 to z_i, respectively $(0\leqq i\leqq k-1)$.

(M) Suppose that y_i is a vertex of $C''[i, x_i]$ different from x_i, for some $i\neq k-1$ (Figure 18).

Set $L=C''[x_1, y_i]\cup Q_i\cup C''[x_i, 1]\cup P_i$.

Then $D=(L\cup P_0\cup C''[x_0, x_1])+(L\cup P_1)$.

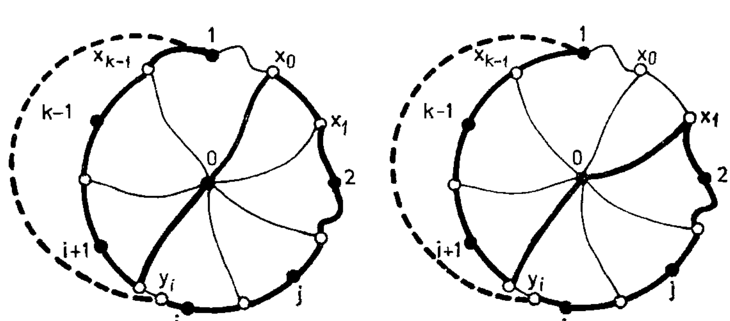

Fig. 18

(N) Suppose that y_i is a vertex of $C''[x_i, i+1]$ different from x_i, for some $i \neq 1, k-1$ (Figure 19).

Set $L = C''[1, x_0] \cup Q_i \cup C''[y_i, z_{k-1}] \cup R_{k-1} \cup C''[2, x_i] \cup P_i$.

Then $D = (L \cup P_0) + (L \cup P_1 \cup C''[x_0, x_1])$.

In view of (M) and (N), we may assume that $y_i = x_i$ and, similarly, that $z_i = x_i$ $(2 \leqq i \leqq k-1)$. Thus

$$X = Y = Z. \tag{11}$$

Consider, next, some vertex j where $3 \leqq j \leqq k-1$. Set

$$G''' = \bigcup_{x \in X} P_x \bigcup_{y \in Y} Q_y \bigcup_{z \in Z} P_z \cup C''[x_j, x_{j-1}].$$

Then (by the Lemma), j is linked by independent (j, G''')-paths to a set W of k vertices of G''', where $\{x_{j-1}, x_j\} \subseteq W$. Denote the paths linking j to x_{j-1} and x_j by S_{j-1} and S_j, respectively, and set $C''' = S_{j-1} \cup S_j \cup C''[x_j, x_{j-1}]$. Let $w \in W \setminus \{x_{j-1}, x_j\}$ and denote by S_w the path linking j and w. Note that, by (11), we may assume that $w \notin \{0, 1, 2\}$ and that w is not an internal vertex of any P_x, Q_y or R_z.

(O) Suppose that w is a vertex of $C'''[i, x_i]$ different from x_i $(i \neq 1, j)$. We may assume, by symmetry, that $i > j$ (Figure 20).

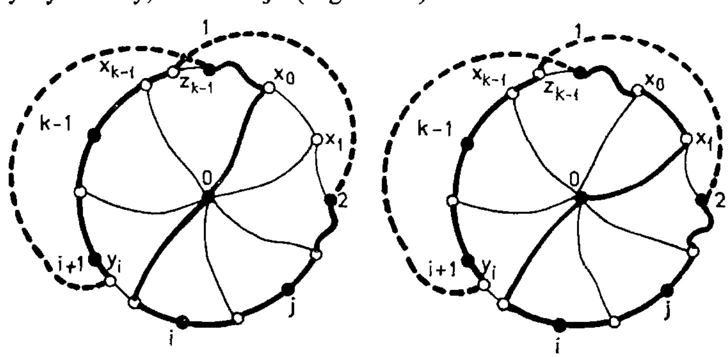

Fig. 19

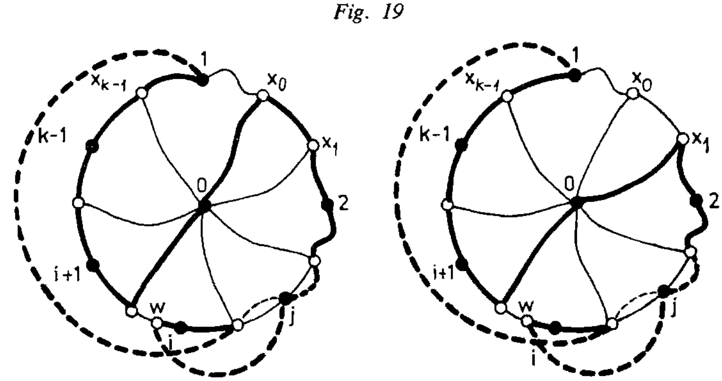

Fig. 20

Set $L = C'''[x_1, j] \cup S_w \cup C'''[x_j, w] \cup Q_j \cup C'''[x_i, 1] \cup P_i$.

Then $D = (L \cup P_0 \cup C'''[x_0, x_1]) + (L \cup P_1)$.

(P) Suppose that w is a vertex of $C'''[x_i, i+1]$ different from x_i ($i \neq 1, j$). As before, we may assume that $i > j$ (Figure 21).

Set $L = C'''[x_1, j] \cup S_w \cup C'''[w, 1] \cup Q_j \cup C'''[x_j, x_i] \cup P_i$.

Then $D = (L \cup P_0 \cup C'''[x_0, x_1]) + (L \cup P_1)$.

(Q) Suppose that w is a vertex of $C'''[1, x_0]$ different from x_0 (Figure 22).

Set $L = C'''[x_1, j] \cup S_w \cup C'''[x_j, w] \cup P_j$.

Then $D = (L \cup P_0 \cup C'''[x_0, x_1]) + (L \cup P_1)$.

A similar argument applies if w is a vertex of $C'''[x_1, 2]$ different from x_1.

(R) Suppose that w is an internal vertex of $C'''[x_0, x_1]$ (Figure 23).

Set $L = S_w \cup C'''[j, 1] \cup Q_{j-1} \cup C'''[x_1, x_{j-1}]$.

Then $D = (L \cup C'''[x_0, w] \cup P_0 \cup P_1) + (L \cup C'''[w, x_1])$.

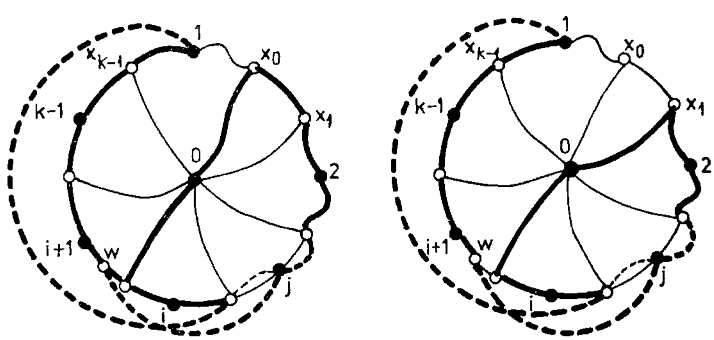

Fig. 21

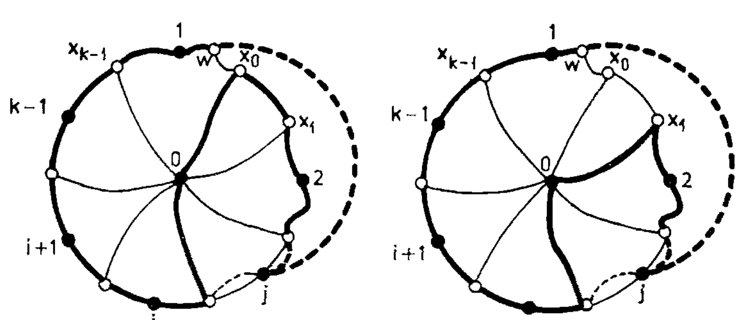

Fig. 22

We may therefore assume that none of the cases (O)—(R) applies. It follows that $W=X=Y=Z$.

Applying this argument to each vertex j $(3\leqq j\leqq k-1)$, we observe that, by (O)—(Q), the paths so defined must be internally-disjoint. Therefore G contains a subdivision F of $K_{k,k}+e$ in which the two colour-classes of $K_{k,k}$ are the sets S and X, and the edge e joins two vertices (x_0 and x_1) of X. Denote the path connecting i and x_j in F by P_{ij} and the path connecting x_0 and x_1 by P. Figure 24 depicts the case $k=4$.

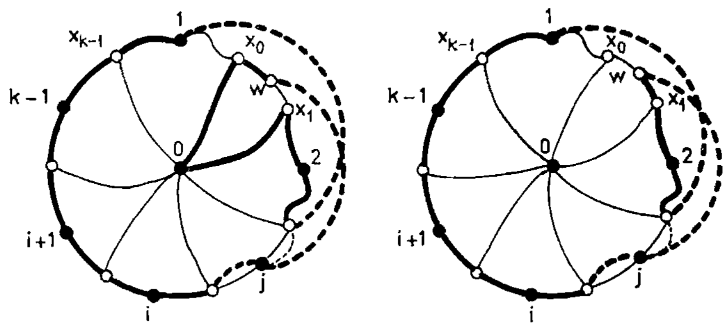

Fig. 23

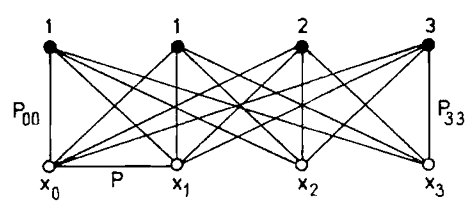

Fig. 24

Suppose that condition (ii) of the theorem does not hold, so that there is a path in $G-X$ connecting two vertices p and q of S. It follows that there is a path Q, disjoint from X and internally-disjoint from F, which starts at a vertex u of P_{pr} and ends at a vertex v of P_{qs} or P, for some r, s.

In order to reduce the number of cases to be investigated, we note that each 4-cycle of $K_{k,k}$ can be expressed as the sum of two Hamilton cycles. Therefore, for $1\leqq i\leqq k-1$, the cycle $P_{00}\cup P_{i0}\cup P_{i1}\cup P_{01}$ can be expressed as the sum of two cycles through S. Since

$$D = P_{00}\cup P\cup P_{01} = (P_{00}\cup P_{i0}\cup P_{i1}\cup P_{01})+(P_{i0}\cup P\cup P_{i1})$$

it suffices to prove that, for some i $(0\leqq i\leqq k-1)$, the cycle $P_{i0}\cup P\cup P_{i1}$ is expressible as the sum of an even number of cycles through S, unless $k=3$. Therefore, the vertices of S are, for our purposes, indistinguishable. By the same token, we may regard both the vertices of $X\setminus\{x_0, x_1\}$ and the vertices of $\{x_0, x_1\}$ as indistinguishable.

For ease of exposition and so as to simplify the diagrams, we shall just consider the cases $k=3, 4$; the extension from the case $k=4$ to $k>4$ will be clear.

Suppose, first, that $k=4$. Set

$$\alpha = |\{r, s\}|, \quad \beta = |\{r, s\} \cap \{0, 1\}|, \quad \gamma = |\{r\} \cap \{0, 1\}|$$

If $v \notin V(P)$, there are five possibilities, depending on the values of α and β; if $v \in V(P)$, two further possibilities arise, depending on the value of γ.

Case 1: $\alpha=1, \beta=0$. Let $p=1, q=2, r=s=2$ (Figure 25).

Set $L=P_{02} \cup P_{32} \cup P_{33} \cup P_{23} \cup P_{22}[2, v] \cup Q \cup P_{12}[1, u] \cup P_{10}$.

Then $D=(L \cup P_{00})+(L \cup P \cup P_{01})$.

Case 2: $\alpha=1, \beta=1$. Let $p=1, q=2, r=s=1$ (Figure 26).

Set $L=P_{02} \cup P_{32} \cup P_{33} \cup P_{23} \cup P_{21}[2, v] \cup Q \cup P_{11}[1, u] \cup P_{10}$.

Then $D=(L \cup P_{00})+(L \cup P \cup P_{01})$.

Case 3: $\alpha=2, \beta=0$. Let $p=2, q=3, r=3, s=2$ (Figure 27).

Set $L=P_{02} \cup P_{22} \cup P_{23}[2, u] \cup Q \cup P_{32}[3, v] \cup P_{33} \cup P_{13} \cup P_{11}$.

Then $D=(L \cup P \cup P_{00})+(L \cup P_{01})$.

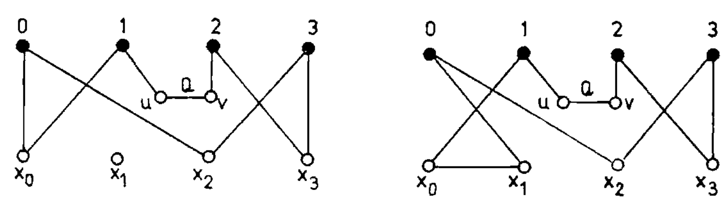

Fig. 25

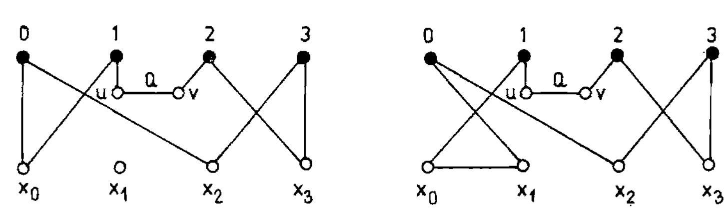

Fig. 26

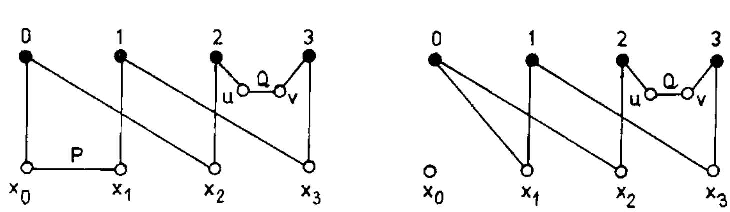

Fig. 27

Case 4: $\alpha=2, \beta=1$. Let $p=1, q=2, r=2, s=1$ (Figure 28).
Set $L=P_{02}\cup P_{32}\cup P_{33}\cup P_{23}\cup P_{21}[2, v]\cup Q\cup P_{12}[1, u]\cup P_{11}$.
Then $D=(L\cup P\cup P_{00})+(L\cup P_{01})$.

Case 5: $\alpha=2, \beta=2$. Let $p=0, q=1, r=0, s=1$ (Figure 29).
Set $L=P_{23}\cup P_{33}\cup P_{32}\cup P_{12}\cup P_{11}[1, v]\cup Q\cup P_{00}[0, u]\cup P_{01}$.
Then $P_{20}\cup P\cup P_{21}=(L\cup P\cup P_{20})+(L\cup P_{21})$.

Case 6: $\gamma=0$. Let $p=0, r=2$ (Figure 30).
Set $L=P_{10}\cup P_{12}\cup P_{32}\cup P_{33}\cup P_{23}\cup P_{21}\cup Q\cup P_{02}[0, u]$.
Then $D=(L\cup P_{00}\cup P[v, x_1])+(L\cup P_{01}\cup P[x_0, v])$.

Case 7: $\gamma=1$. Let $p=0, r=1$ (Figure 31).
Set $L=P_{21}\cup P_{23}\cup P_{33}\cup P_{32}\cup P_{01}[0, u]\cup Q\cup P_{10}$.
Then $D=(L\cup P_{00}\cup P_{12}\cup P[v, x_1])+(L\cup P_{02}\cup P_{11}\cup P[x_0, v])+(P_{01}\cup P_{11}\cup P_{12}\cup P_{02})$

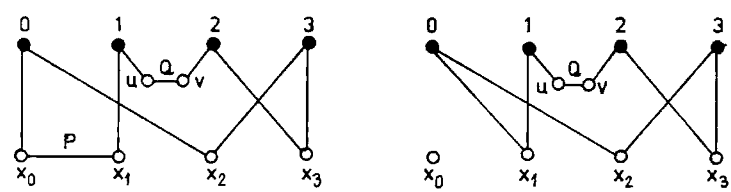

Fig. 28

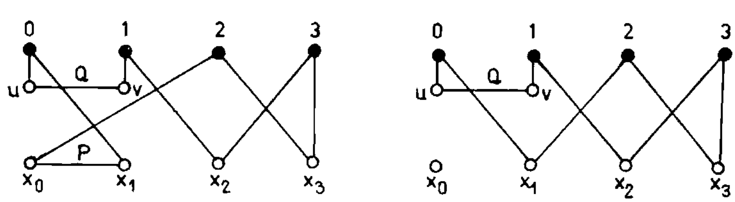

Fig. 29

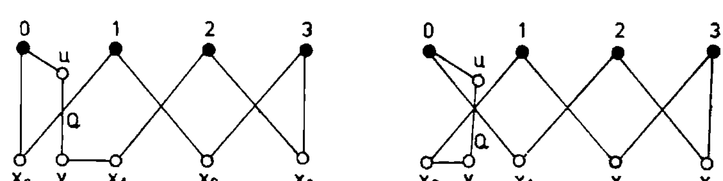

Fig. 30

and, as noted above, the cycle $P_{01} \cup P_{11} \cup P_{12} \cup P_{02}$ can be expressed as the sum of two cycles through S.

This establishes the theorem in the case $k \geqq 4$. So suppose, now, that $k=3$. Of the above seven cases, all but case 3 (which cannot arise) must be examined. With the exception of case 1, a simple modification of the cycles described for the case $k=4$ (replacing $P_{23} \cup P_{33} \cup P_{32}$ by P_{22}) results in appropriate cycles for the case $k=3$. So it is only case 1 which needs to be examined in detail. Set $F'=F \cup Q$.

(S) Suppose that the path P has length at least two. Then, because G is 3-connected, there exists a path R, internally-disjoint from F', between an internal vertex w of P and some other vertex of F'. By cases 6 and 7, we may assume that this other vertex is x_2 (Figure 32).

Set $L=P_{21} \cup P_{22}[2, v] \cup Q \cup P_{12}[1, u] \cup P_{10} \cup R \cup P_{02}$.

Then $D=(L \cup P_{00} \cup P[w, x_1])+(L \cup P_{01} \cup P[x_0, w])$.

So we may now assume that P is the edge x_0x_1. If, in $G-\{x_0, x_1\}$, each of 0, 1, 2 is separated from the other two by a cut vertex, G has the structure described in condition (iii) of the theorem. If not, we may suppose, without loss of generality, that 2 is not separated from 0 and 1 by a cut vertex in $G-\{x_0, x_1\}$. Set

$$F \quad = P_{00} \cup P_{01} \cup P_{02} \cup P_{10} \cup P_{11} \cup P_{12}.$$

Then 2 is connected by two independent paths to F'' in $G-\{x_0, x_1\}$. In other words, there is a path P', disjoint from $\{x_0, x_1\}$ and internally-disjoint from F'', which includes 2 and connects two vertices x, y of F''. Because of symmetry between 0 and 1 and between x_0 and x_1, there are five cases to be considered. We demonstrate that, in each case, D can be expressed as the sum of an even number of cycles of type at least (2, 1). Set $Q'=P_{02} \cup P_{12}$.

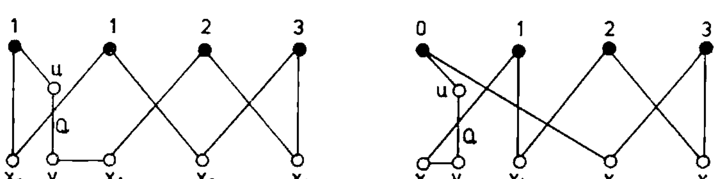

Fig. 31

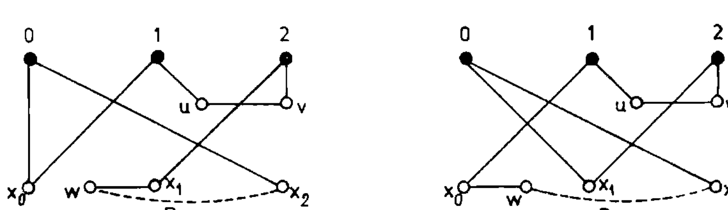

Fig. 32

Case 1.1: $x, y \in Q'$ (Figure 33).

Set $L = Q'[0, x] \cup P' \cup Q'[y, 1] \cup P_{11}$.

Then $D = (L \cup x_0 x_1 \cup P_{00}) + (L \cup P_{01})$.

Case 1.2: $x, y \in P_{11}$ (Figure 34).

Set $L = Q' \cup P_{11}[1, x] \cup P' \cup P_{11}[y, 1]$.

Then $D = (L \cup x_0 x_1 \cup P_{00}) + (L \cup P_{01})$.

Case 1.3: $x \in Q'$, $y \in P_{10}$ (Figure 35).

Set $L = Q'[0, x] \cup P' \cup P_{10}[1, y] \cup P_{11}$.

Then $D = (L \cup x_0 x_1 \cup P_{00}) + (L \cup P_{01})$.

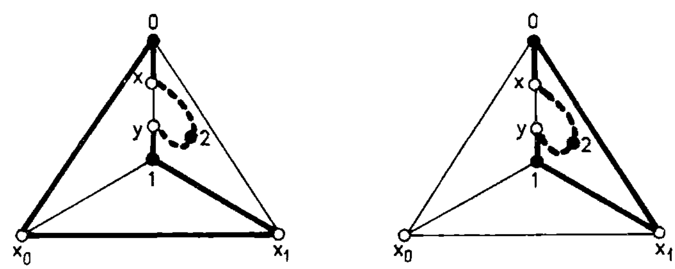

Fig. 33

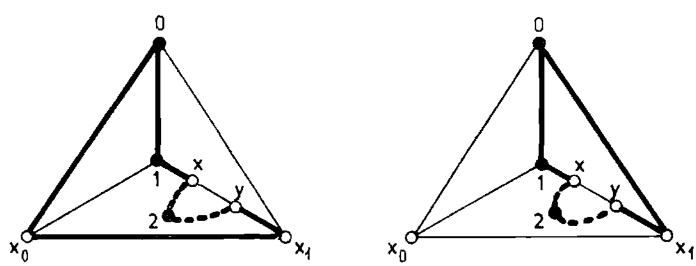

Fig. 34

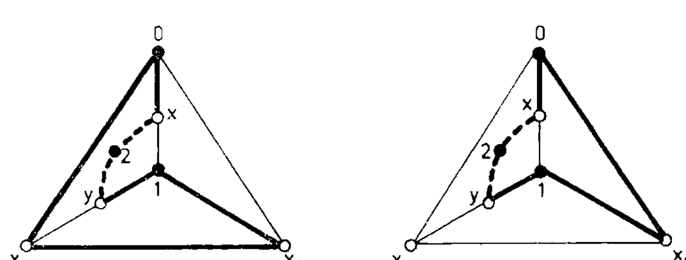

Fig. 35

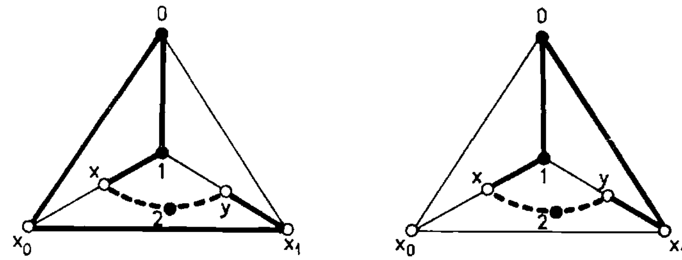

Fig. 36

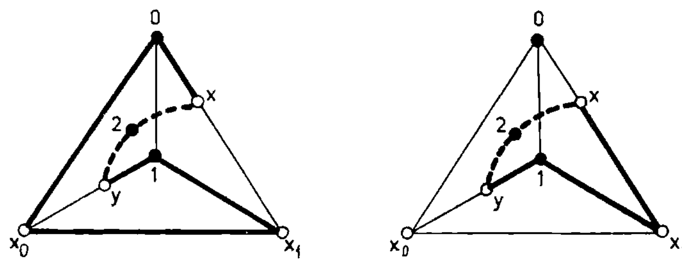

Fig. 37

Case 1.4: $x \in P_{10}$, $y \in P_{11}$ (Figure 36).

Set $L = Q' \cup P_{10}[1, x] \cup P' \cup P_{11}[y, x_1]$.

Then $D = (L \cup x_0x_1 \cup P_{00}) + (L \cup P_{01})$.

Case 1.5: $x \in P_{01}$, $y \in P_{10}$ (Figure 37).

Set $L = P' \cup P_{10}[1, y] \cup P_{11}$.

Then $D = (L \cup x_0x_1 \cup P_{00} \cup P_{01}[0, x]) + (L \cup P_{01}[x, x_1])$.

This completes the proof of Theorem 2.

5. Possible generalizations

It is natural to ask whether our results can be strengthened by considering the cycles through a set of edges rather than vertices. Lovász [7] and Woodall [18] have conjectured that Corollary 3 can be so strengthened. Some more-or-less trivial assumptions must be made here. Clearly, the given edges must form one or more disjoint paths. Also, if k is odd, they cannot be $k-1$ edges of a k-element coboundary. Whether or not these conditions are sufficient remains to be decided. Partial results have been obtained by Woodall [18] and Thomassen [12].

A more general question is the following:

Problem. Let G be a k-connected graph and let $F_1, F_2, \ldots, F_k$ be sets of edges of G. When does there exist a cycle C in G such that $|F_i \cap C|$ is odd for $i=1, 2, \ldots, k$?

A trivial necessary condition is that no F_i should be a coboundary; more generally, no coboundary should be the sum of an odd number of sets F_i. It is not difficult to see that these conditions are necessary and sufficient for the existence of an eulerian subgraph H (that is, a member of the cycle space) such that $|H \cap F_i|$ is odd for each i. However, if H is required to be a cycle, further necessary conditions must be imposed; for instance, the sets F_i with cardinality one should form disjoint paths or a single cycle. Again, we do not know if these conditions are sufficient.

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Added in proof. Recently Häggkvist and Thomassen proved the conjecture of Lovász and Woodall.

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